

ON A THEOREM OF STAFFORD

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ABSTRACT. In [6] Stafford proved that every left or right ideal of the Weyl algebra $A_n(K) = K[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ (K a field of characteristic zero) is generated by two elements. Consider the ring $D_n := K[[x_1, \dots, x_n]]\langle \partial_1, \dots, \partial_n \rangle$ of differential operators over the ring of formal power series $K[[x_1, \dots, x_n]]$. In this paper we prove that every left or right ideal of the ring $E_n := K((x_1, \dots, x_n))\langle \partial_1, \dots, \partial_n \rangle$ of differential operators over the field of formal Laurent series $K((x_1, \dots, x_n))$ is generated by two elements. The same is true for the ring of differential operators over the convergent Laurent series $\mathbb{C}\{\{x_1, \dots, x_n\}\}$. This is in accordance with the conjecture that says that in a (noncommutative) noetherian simple ring, every left or right ideal is generated by two elements.

1. INTRODUCTION

In [6] Stafford proved that every left or right ideal of the Weyl algebra $A_n(K) = K[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ (K a field of characteristic zero) is generated by two elements. It would be interesting to have a similar result for the ring $D_n := K[[x_1, \dots, x_n]]\langle \partial_1, \dots, \partial_n \rangle$ of differential operators over the ring of formal power series $K[[x_1, \dots, x_n]]$. In this paper we prove that every left or right ideal of the ring $E_n := K((x_1, \dots, x_n))\langle \partial_1, \dots, \partial_n \rangle$ of differential operators over the field of formal Laurent series $K((x_1, \dots, x_n))$ is generated by two elements. The same is true for the ring of differential operators over the convergent Laurent series $\mathbb{C}\{\{x_1, \dots, x_n\}\}$ (this is the field of fractions of the domain \mathcal{O}_n of germs of convergent complex power series around the origin of \mathbb{C}^n). Note that this is in accordance with the conjecture that says that in a (noncommutative) noetherian simple ring, every left or right ideal is generated by two elements.

The main difficulty here is that the symmetry between the x 's and the ∂ 's, present in the Weyl algebra and which is fundamental in the prove of Stafford's theorem, is broken in the ring D_n . Since we are working over the base ring of power series, infinite sums in the x 's are permitted while only finite sums in

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the ∂ 's occur. To deal with this problem we appeal to Weierstrass' preparation theorem to put an element $v \in D_n$ in an appropriate form (see lemma 4.1). In general, however, we follow the proof of Stafford's theorem in Bjork's book ([1]) and we do the necessary modifications to deal with the power series case.

2. BASIC NOTATIONS

Let K be a field of characteristic zero and T be a skew field of characteristic zero. By $T[[x]]$ we denote the ring of power series in one indeterminate with coefficients in T and by $T((x))$ its quotient skew field. In paragraph 3 we will be interested in the ring $S = T((x))\langle\partial_x\rangle$, the ring of differential operators with coefficients in $T((x))$. This paragraph is mostly concerned with free S -modules of finite rank over S .

In paragraph 4 the following notation will be used. For each $0 \leq r \leq n$ let $D_r = K[[x_1, \dots, x_r]]\langle\partial_1, \dots, \partial_r\rangle$ be the ring of differential operators over the ring of formal power series $K[[x_1, \dots, x_r]]$ and let F_r be its quotient ring. F_r exists because D_r is an Ore domain. Since x_{r+1}, \dots, x_n commute with the elements in F_r we also get the division ring $F_r((x_{r+1}, \dots, x_n))$ which by definition is the quotient ring of $D_r[[x_{r+1}, \dots, x_n]]$. Also, for each $0 \leq r \leq n$, we put $R_r = F_r((x_{r+1}, \dots, x_n))\langle\partial_{r+1}, \dots, \partial_n\rangle$. Of course if $r = n$, then $R_n = F_n$.

3. THE RING $S = T((x))\langle\partial_x\rangle$

Let us explore some properties of $S = T((x))\langle\partial_x\rangle$, the ring of linear differential operators with coefficients in rational expressions in x over the skew field T . Recall that S is a noncommutative noetherian simple ring.

Suppose that F is a field contained in T . By $F[[x]]\langle\partial_x\rangle$ we denote the ring of differential operators with coefficients in the ring of power series $F[[x]]$.

Lemma 3.1. *Let $0 \neq \alpha \in S$. Then the S -module $S/S\alpha$ has finite length.*

Proof. It follows immediately from the division algorithm on S . See also [1] lemma 8.8, page 27. \square

Lemma 3.2. *Let $\delta_1, \dots, \delta_m$ be a set of non zero elements in $F[[x]]\langle\partial_x\rangle \subset S$. Let $0 \neq \alpha \in S$ and $S^{(m)} = S\varepsilon_1 + \dots + S\varepsilon_m$ be a free S -module of rank m with basis $\varepsilon_1, \dots, \varepsilon_m$. Let M be the S -submodule of $S^{(m)}$ generated by the set $\{\alpha\delta_1 f \varepsilon_1 + \dots + \alpha\delta_m f \varepsilon_m / f \in \mathbb{Z}[[x]]\langle\partial_x\rangle\}$. Then $M = S^{(m)}$.*

Proof. To simplify the notation we put $\partial := \partial_x$

Let us first observe that both the assumption and the conclusion are unchanged if the m -tuple $\delta_1, \dots, \delta_m$ is replaced by an m -tuple β_1, \dots, β_m , where $\beta_i = \sum a_{ij} \delta_j$ and (a_{ij}) is an $m \times m$ invertible matrix with $a_{ij} \in F$. Of course, while we replace $\delta_1, \dots, \delta_m$ by β_1, \dots, β_m under a F -linear transformation (a_{ij}) we also replace the free generators $\varepsilon_1, \dots, \varepsilon_m$ of $S^{(m)}$ by ζ_1, \dots, ζ_m where $\zeta_i = \sum b_{ij} \varepsilon_j$, $(b_{ij}) = (a_{ij})^{-1}$.

Let $\text{ord}(\delta_i)$ be the ∂ -order of δ_i . We can assume that these ∂ -orders decrease, ie, $\text{ord}(\delta_1) \geq \text{ord}(\delta_2) \geq \dots \geq \text{ord}(\delta_m)$. Hence there exists an integer ω and some $1 \leq l \leq m$ such that $\omega = \text{ord}(\delta_1) = \dots = \text{ord}(\delta_l)$, while $\text{ord}(\delta_i) < \omega$ if $i > l$.

If $1 \leq i \leq l$ we can write $\delta_i = r_i + p_i(x)\partial^\omega$, where $\text{ord}(r_i) < \omega$ and $p_i(x) \in F[[x]]$. We can assume that $\text{val}(p_1) \leq \text{val}(p_2) \leq \dots \leq \text{val}(p_l)$, where $\text{val}(p_i)$ is the usual valuation of the power series p_i . If $\text{val}(p_1) = \text{val}(p_2) = \mu$, then there exists some $t \in F$ such that $\text{val}(p_2 - tp_1) > \mu$. Replace δ_2 by $\delta_2 - t\delta_1$ while $\delta_1, \delta_3, \dots, \delta_m$ are unchanged. Then, after some F -linear transformations we can assume that $\text{val}(p_1) < \text{val}(p_2) < \dots < \text{val}(p_l)$. With these normalizations in hand we begin to prove that $\varepsilon_1 \in M$.

Let $k = \text{ord}(\alpha)$. Since $S = T((x))\langle\partial\rangle$ we can assume that $\alpha = \alpha_0 + \partial^k$ where $\text{ord}(\alpha_0) < k$. If $1 \leq i \leq l$ we have $\alpha\delta_i = p_i(x)\partial^{k+\omega} + \psi_i$ where $\text{ord}(\psi_i) < k + \omega$ and if $l < i \leq m$, $\text{ord}(\alpha\delta_i) < k + \omega$.

Now if $g \in S$ we put $g_1 = [g, x] = gx - xg$ the commutator of g and x . Inductively, let $g_{\nu+1} = [g_\nu, x]$. The element g_ν is called the ν -fold commutator of g and x . The ν -fold commutator of ∂^ν and x is $\nu!$ for all positive integers ν , while the ν -fold commutator of ∂^s and x is zero if $s < \nu$.

If we apply this to the elements $\alpha\delta_1, \dots, \alpha\delta_m$ we see that the $(k + \omega)$ -fold commutator of $\alpha\delta_i$ and x is $p_i(x)(k + \omega)!$ for all $1 \leq i \leq l$, while they are zero if $l < i \leq m$.

The definition of M implies that M is stable under the ν -fold commutator with x , i.e., if $m \in M$ then the ν -fold commutator of m and x is in M for all positive integers ν . In fact, note that if $f \in \mathbb{Z}[[x]]\langle\partial\rangle$, then $[\alpha\delta f, x] = \alpha\delta(fx) - x(\alpha\delta f)$.

For $f = 1 \in \mathbb{Z}[[x]]\langle\partial\rangle$ we have that $a = \alpha\delta_1\varepsilon_1 + \dots + \alpha\delta_m\varepsilon_m \in M$. If v_1 is the $(k + \omega)$ -fold commutator of a and x divided by $(k + \omega)!$, then $v_1 \in M$ and

$$v_1 = p_1(x)\varepsilon_1 + \dots + p_l(x)\varepsilon_l.$$

Now, if $h \in S$, let $h_1 = [h, \partial] = h\partial - \partial h$ be the commutator of h with ∂ . Inductively, we define $h_{\nu+1}$ by $h_{\nu+1} = [h_\nu, \partial]$. The element h_ν is called the ν -fold commutator of h and ∂ . We observe that M is stable under the ν -fold commutator with ∂ for all positive integers ν . In fact, note that if $f \in \mathbb{Z}[[x]]\langle\partial\rangle$, then $[\alpha\delta f, \partial] = \alpha\delta(f\partial) - \partial(\alpha\delta f)$.

Since $v_1 \in M$ we have that if $v_1^{(\mu)}$ is the μ -fold commutator of v_1 and ∂ , where $\mu = \text{val}(p_1(x)) < \dots < \text{val}(p_l(x))$, then $v_1^{(\mu)}$ is in M and

$$v_1^{(\mu)} = p_1^{(\mu)}(x)\varepsilon_1 + p_2^{(\mu)}(x)\varepsilon_2 + \dots + p_l^{(\mu)}(x)\varepsilon_l \in M,$$

where $p^{(\mu)}(x)$ denotes the usual μ -derivative of a power series $p(x)$. Note that $u(x) := p_1^{(\mu)}(x)$ is a unit in $F[[x]]$ and $\text{val}(p_j^{(\mu)}(x)) = \text{val}(p_j(x)) - \mu > 0$, $j = 2, \dots, l$.

Define $v_2 := v_1 - p_1(x)(u(x))^{-1}v_1^{(\mu)}$. Then $v_2 \in M$ and

$$v_2 = q_2(x)\varepsilon_2 + \dots + q_l(x)\varepsilon_l,$$

where $q_j(x) = p_j(x) - p_1(x)(u(x))^{-1}p_j^{(\mu)}(x)$, $j = 2, \dots, l$. A simple calculation shows that $q_j(x)$ is non-zero with $\text{val}(q_j(x)) = \text{val}(p_j(x))$, for all $j = 2, \dots, l$. Therefore $\text{val}(q_2) < \text{val}(q_3) < \dots < \text{val}(q_l)$.

We now, repeat the previous argument using the commutator of v_2 and ∂ until we get $v_3 = r_3(x)\varepsilon_3 + \dots + r_l(x)\varepsilon_l \in M$. Proceeding in this way we finally get $v_l(x) = \tilde{u}(x)\varepsilon_l \in M$, where $\tilde{u}(x)$ is a unit in $F[[x]]$. Therefore $\varepsilon_l \in M$.

If $l = 1$, we are done. If $l > 1$, since $v_{l-1} \in M$, we have that $\varepsilon_{l-1} \in M$. Going backwards we get $\varepsilon_1 \in M$.

Restricting the attention to the $(m-1)$ -tuple $\delta_2, \dots, \delta_m$ and the S -module $S^{(m-1)} = S\varepsilon_2 + \dots + S\varepsilon_m$, the lemma follows by induction over m . \square

Lemma 3.3. *Let $\delta_1, \dots, \delta_m$ be a set of non zero elements in $F[[x]]\langle\partial_x\rangle \subset S$ and let M be a S -submodule of $S^{(m)}$ such that the S -module $S^{(m)}/M$ has finite length. If $0 \neq \alpha \in S$, there exists some $f \in \mathbb{Z}[[x]]\langle\partial_x\rangle$ such that*

$$S^{(m)} = M + S(\alpha\delta_1 f\varepsilon_1 + \dots + \alpha\delta_m f\varepsilon_m).$$

Proof. See [1], Lemma 8.10, page 28. \square

Corollary 3.4. *Let $0 \neq \rho \in S$ and let $\delta_1, \dots, \delta_m$ be a set of non zero elements in $F[[x]]\langle\partial_x\rangle \subset S$. Then there exists some $f \in \mathbb{Z}[[x]]\langle\partial_x\rangle$ such that*

$$S^{(m)} = S^{(m)}\rho + S(\rho\delta_1 f\varepsilon_1 + \dots + \rho\delta_m f\varepsilon_m)$$

Proof. See [1], Corollary 8.11, page 29. \square

Lemma 3.5. *Let $\delta_1, \dots, \delta_m$ be a set of non zero elements in $F[[x]]\langle\partial_x\rangle \subset S$ and let $0 \neq \rho \in S$. Consider the free S -module $S^{m+1} = S\varepsilon_0 + S\varepsilon_1 + \dots + S\varepsilon_m$ with basis $\varepsilon_0, \dots, \varepsilon_m$. Then there exists some $f \in \mathbb{Z}[[x]]\langle\partial_x\rangle$ such that*

$$S^{(m+1)} = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 f\varepsilon_1 + \dots + \delta_m f\varepsilon_m)$$

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Proof. See [1], Lemma 8.13, page 29. \square

4. LEMMAS FOR D_r AND R_r

In lemma 4.4 we will need to apply Weierstrass' preparation theorem to an non-zero element $v \in D_n = K[[x_1, \dots, x_n]]\langle\partial_1, \dots, \partial_n\rangle$. We will then state a separate lemma to prepare this element.

Lemma 4.1. *Let $v \in D_n$ be a non zero element. For any r , $0 \leq r \leq n-1$, v can be written in the following form:*

$$v = \omega_1\beta_1 G_1 + \dots + \omega_m\beta_m G_m,$$

where $\omega_1, \dots, \omega_m \in K[[x_1, \dots, x_n]]$ are units, $\beta_1, \dots, \beta_m \in K[[x_{r+1}]]\langle\partial_{r+1}\rangle$ and $G_1, \dots, G_m \in D(r+1) := K[[x_1, \dots, x_{r+1}, \dots, x_n]]\langle\partial_1, \dots, \partial_{r+1}, \dots, \partial_n\rangle$.

Proof. Since $v \in D_n$ is a non zero element, we can write v as a finite sum $\sum_{\alpha} p_{\alpha}(x_1, \dots, x_n) \partial^{\alpha}$, where each $p_{\alpha}(x_1, \dots, x_n) \in K[[x_1, \dots, x_n]]$ and $\partial^{\alpha} = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. Let k_{α} be the order of the series p_{α} . After a suitable linear change of variables we can assume that $p_{\alpha}(0, \dots, 0, x_{r+1}, 0, \dots, 0) \in K[[x_{r+1}]]$ is non zero for all α and has valuation k_{α} as a series in the variable x_{r+1} . In fact, since K is an infinite field and the p_{α} are in a finite number there is a single linear change of variables that works for all p_{α} . (See [2], lemma 2, page 17 and the remark that follows this lemma).

Now let us fixed α . Then the series $p_{\alpha}(x_1, \dots, x_n)$ can be written as $\sum_{i=0}^{\infty} q_i(x_1, \dots, x_{r+1}, \dots, x_n) x_{r+1}^i$, where $q_i(x_1, \dots, x_{r+1}, \dots, x_n) \in K[[x_1, \dots, x_{r+1}, \dots, x_n]]$. By our assumptions, $q_{k_{\alpha}}(x_1, \dots, x_{r+1}, \dots, x_n)$ is a unit in $K[[x_1, \dots, x_{r+1}, \dots, x_n]]$. By Weierstrass' preparation theorem (see [4], page 208) we can write

$$p_{\alpha} = u(b_0 + b_1 x_{r+1} + \dots + x_{r+1}^{k_{\alpha}}),$$

where u is a unit in $K[[x_1, \dots, x_n]]$ and $b_j \in K[[x_1, \dots, x_{r+1}, \dots, x_n]]$. Therefore we have

$$\begin{aligned} p_{\alpha} \partial^{\alpha} &= u(b_0 + b_1 x_{r+1} + \dots + x_{r+1}^{k_{\alpha}}) \partial_1^{\alpha_1} \dots \partial_{r+1}^{\alpha_{r+1}} \dots \partial_n^{\alpha_n} = u(\partial_{r+1}^{\alpha_{r+1}})(b_0 \partial_1^{\alpha_1} \dots \partial_{r+1}^{\alpha_{r+1}-1} \dots \partial_n^{\alpha_n}) + \\ &u(x_{r+1} \partial_{r+1}^{\alpha_{r+1}})(b_1 \partial_1^{\alpha_1} \dots \partial_{r+1}^{\alpha_{r+1}-1} \dots \partial_n^{\alpha_n}) + \dots + u(x_{r+1}^{k_{\alpha}} \partial_{r+1}^{\alpha_{r+1}})(\partial_1^{\alpha_1} \dots \partial_{r+1}^{\alpha_{r+1}-1} \dots \partial_n^{\alpha_n}), \end{aligned}$$

which has the desired form. Since $v = \sum_{\alpha} p_{\alpha} \partial^{\alpha}$ we are done. \square

The following lemmas 4.3 and 4.4 are equivalent. We will then prove just the second. To prove it we will need the following result.

Lemma 4.2. *Let $0 \leq r \leq n-1$ and let q be a non zero element of $D_r[[x_{r+1}, \dots, x_n]]$ and let a_1, \dots, a_t be a finite set in D_n . Then there exists some $\rho \in D_r[[x_{r+1}, \dots, x_n]]$, $\rho \neq 0$, such that $\rho a_j \in D_n q$, for each $j = 1, \dots, t$.*

Proof. See [1], lemma 8.5, page 26. \square

Lemma 4.3. *Let $0 \leq r \leq n-1$ and let $0 \neq q \in D_{r+1}[[x_{r+2}, \dots, x_n]]$. If u and v are two elements in D_n with $v \neq 0$, then there exists $f \in D_n$ and $Q_r \in D_r[[x_{r+1}, \dots, x_n]]$ such that*

$$Q_r \in D_n q + D_n(u + v f).$$

Recall that $R_r = F_r((x_{r+1}, \dots, x_n)) \langle \partial_{r+1}, \dots, \partial_n \rangle$, where F_r is the quotient ring of $D_r = K[[x_1, \dots, x_r]] \langle \partial_1, \dots, \partial_r \rangle$.

Lemma 4.4. *Let $0 \leq r \leq n-1$ and let $0 \neq q \in D_{r+1}[[x_{r+2}, \dots, x_n]]$ and let u and $v \in D_n$ with $v \neq 0$. Then there exist some $f \in D_n$ such that*

$$R_r = R_r q + R_r(u + v f).$$

Proof. Let us fix r such that $0 \leq r \leq n-1$. Since $v \neq 0$, by lemma 4.1 we can write

$$v = \omega_1 \beta_1 G_1 + \dots + \omega_m \beta_m G_m,$$

where $\omega_1, \dots, \omega_m \in K[[x_1, \dots, x_n]]$ are units, $\beta_1, \dots, \beta_m \in K[[x_{r+1}]] \langle \partial_{r+1} \rangle$ and $G_1, \dots, G_m \in D(r+1) = K[[x_1, \dots, x_{r+1}, \dots, x_n]] \langle \partial_1, \dots, \partial_{r+1}, \dots, \partial_n \rangle$.

Let $\delta_i = \omega_i \beta_i$, for each $1 \leq i \leq m$. Then $\delta_i \in K[[x_1, \dots, x_n]] \langle \partial_{r+1} \rangle$. To apply lemma 3.5, we observe that $K[[x_1, \dots, x_n]] = (K[[x_1, \dots, x_{r+1}, \dots, x_n]])[[x_{r+1}]] \subset F[[x_{r+1}]]$, where F is the quotient field of $K[[x_1, \dots, x_{r+1}, \dots, x_n]]$. Therefore $\delta_i \in F[[x_{r+1}]] \langle \partial_{r+1} \rangle$, F a field of characteristic zero. If $T = F_r((x_{r+1}, \dots, x_n))$, then T is a skew field and $F \subset T$.

Since $v \neq 0$, we have that some $G_i \neq 0$. The ring $D(r+1)$ is simple, which implies the 2-sided ideal generated by G_1, \dots, G_m is the whole ring $D(r+1)$. This gives finite sets a_1, \dots, a_l and b_1, \dots, b_l in $D(r+1)$ such that

$$1 = \sum_{j=1}^m \sum_{\nu=1}^l b_\nu G_j a_\nu$$

and hence $D(r+1) = \Sigma \Sigma D(r+1) G_j a_\nu$. Identifying $D(r+1)$ with a subring of R_r we conclude that $R_r = \Sigma \Sigma R_r G_j a_\nu$.

At this stage we need the following

Claim: To each m -tuple B_1, \dots, B_m in $D(r+1)$ there exists some $f \in \mathbb{Z}[[x_{r+1}]] \langle \partial_{r+1} \rangle$ such that

$$R_r q + R_r u + R_r B_1 + \dots + R_r B_m = R_r q + R_r(u + \delta_1 f B_1 + \dots + \delta_m f B_m).$$

In fact, since $0 \neq q \in D_{r+1}[[x_{r+2}, \dots, x_n]]$, it follows from lemma 4.2 that there exists some $0 \neq \rho \in D_{r+1}[[x_{r+2}, \dots, x_n]]$ such that $\rho B_j \in D_n q$ for all $j = 1, \dots, m$ and also $\rho u \in D_n q$.

Let $S = T((x_{r+1})) \langle \partial_{r+1} \rangle$, then $0 \neq \rho \in S$. Using the lemma 3.5 we get some $f \in \mathbb{Z}[[x_{r+1}]] \langle \partial_{r+1} \rangle$ such that $S^{(m+1)} = S^{(m+1)} \rho + S(\varepsilon_0 + \delta_1 f \varepsilon_1 + \dots + \delta_m f \varepsilon_m)$. Since S is a subring of R_r , we have that $R_r^{(m+1)} = R_r^{(m+1)} \rho + R_r(\varepsilon_0 + \delta_1 f \varepsilon_1 + \dots + \delta_m f \varepsilon_m)$.

Considerer the R_r -linear application $\pi : R_r^{(m+1)} \rightarrow R_r$ defined by $\pi(\varepsilon_0) = u$ and $\pi(\varepsilon_j) = B_j$ for each $1 \leq j \leq m$. Then the image of π is $R_r u + R_r B_1 + \dots + R_r B_m \subseteq R_r$; but $\rho u, \rho B_j \in D_n q$. Then we have that $\pi(\rho \varepsilon_0) = \rho \pi(\varepsilon_0) = \rho u \in R_r q$ and $\pi(\rho \varepsilon_j) = \rho \pi(\varepsilon_j) = \rho B_j \in R_r q$ for each $1 \leq j \leq m$. Therefore $\pi(R_r^{(m+1)} \rho) \subseteq R_r q$. Then

$$\begin{aligned} R_r q + R_r u + R_r B_1 + \dots + R_r B_m &= R_r q + \pi(R_r^{(m+1)}) \\ &\subseteq R_r q + \pi(R_r^{(m+1)} \rho + R_r(\varepsilon_0 + \delta_1 f \varepsilon_1 + \dots + \varepsilon_0 + \delta_m f \varepsilon_m)) \\ &\subseteq R_r q + \pi(R_r^{(m+1)} \rho) + R_r(u + \delta_1 f B_1 + \dots + \delta_m f B_m) \\ &\subseteq R_r q + R_r(u + \delta_1 f B_1 + \dots + \delta_m f B_m) \end{aligned}$$

and this proves the claim because the opposite inclusion is clear.

Now we apply the claim to $B_j = G_j a_1, j = 1, \dots, m$. Then, there exists $f_1 \in \mathbb{Z}[[x_{r+1}]] \langle \partial_{r+1} \rangle$ such that

$$R_r q + R_r u + R_r G_1 a_1 + \dots + R_r G_m a_1 = R_r q + R_r(u + \delta_1 f_1 G_1 a_1 + \dots + \delta_m f_1 G_m a_1).$$

Since $f_1 \in \mathbb{Z}[[x_{r+1}]] \langle \partial_{r+1} \rangle$ and $G_j \in D(r+1)$ commute, we have that

$$\begin{aligned} \delta_1 f_1 G_1 a_1 + \dots + \delta_m f_1 G_m a_1 &= \delta_1 G_1 f_1 a_1 + \dots + \delta_m G_m f_1 a_1 = \omega_1 \beta_1 G_1 f_1 a_1 + \dots + \omega_m \beta_m G_m f_1 a_1 \\ &= v f_1 a_1, \text{ since } v = \omega_1 \beta_1 G_1 + \dots + \omega_m \beta_m G_m. \end{aligned}$$

$$R_r q + R_r u + R_r G_1 a_1 + \dots + R_r G_m a_1 = R_r q + R_r(u + v f_1 a_1).$$

Now we apply the claim again with u replaced by $u + vf_1a_1$ and $B_j = G_ja_2, j = 1, \dots, m$. There exists $f_2 \in \mathbb{Z}[[x_{r+1}]]\langle\partial_{r+1}\rangle$ such that $R_rq + R_r(u + vf_1a_1) + \sum R_rG_ja_2 = R_rq + R_r(u + vf_1a_1 + vf_2a_2)$. Using the previous equation we have

$$R_rq + R_ru + \sum R_rG_ja_1 + \sum R_rG_ja_2 = R_rq + R_r(u + vf_1a_1 + vf_2a_2).$$

In the next step we apply the claim with $B_j = G_ja_3, j = 1, \dots, m$ and u replaced by $u + vf_1a_1 + vf_2a_2$. After l steps we have

$$R_rq + R_r(u + vf_1a_1 + \dots + vf_la_l) = R_rq + R_ru + \sum \sum R_rG_ja_\nu = R_r.$$

Hence the lemma follows with $f = f_1a_1 + \dots + f_la_l$. \square

5. THE PRINCIPAL RESULT

Lemma 5.1. *Let a, b and c non zero elements of D_n . For each $0 \leq r \leq n$ there exist $q_r \in D_r[[x_{r+1}, \dots, x_n]], q_r \neq 0$ and $d_r, e_r \in D_n$ such that*

$$q_rc \in D_n(a + d_rc) + D_n(b + e_rc).$$

Proof. With $d_n = 0$ and $e_n = 0$ we see that the statement is true for $r = n$, since D_n is a left Ore domain and $D_na \subset (D_na + D_nb)$.

For $0 \leq r \leq (n - 1)$ the proof is by induction from $r + 1$ to r . For this see the prove that Proposition 7.3(r+1) \Rightarrow Proposition 7.3(r) in the book of Björk [1, page 22], in which the lemma 7.5 should be replaced by our lemma 4.3. \square

Theorem 5.2. *Any left or right ideal in $K((x_1, \dots, x_n))\langle\partial_1, \dots, \partial_n\rangle$ can be generated by two elements. The same is true for the ring $\mathbb{C}\{\{x_1, \dots, x_n\}\}\langle\partial_1, \dots, \partial_n\rangle$.*

Proof. The ring $E_n = K((x_1, \dots, x_n))\langle\partial_1, \dots, \partial_n\rangle$ is a noetherian ring. Therefore, it is enough to show that given $a, b, c \in E_n$ there exists $d, e \in E_n$ such that $c \in E_n(a + dc) + E_n(b + ec)$.

Take $n = 0$ in the previous lemma. Then, there exists $q_0 \in K[[x_1, \dots, x_n]], q_0 \neq 0$ and $d, e \in D_n$ such that $q_0c \in D_n(a + dc) + D_n(b + ec)$. Since $D_n = K[[x_1, \dots, x_n]]\langle\partial_1, \dots, \partial_n\rangle$, then $c \in E_n(a + dc) + E_n(b + ec)$. \square

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